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EQUIVALENCE OF THE INDUCTION SCHEMA AND THE LEAST NUMBER PRINCIPLE FOR OPEN FORMULAS

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Abstract. Let $\mathcal{L}_A$ be the usual language for arithmetic. Let $\varphi(x)$ be an $\mathcal{L}_A$-formula. $\varphi(x)$ may contain free variables distinct from x as parameters. We consider the following two schemata.

$$(I_{\varphi(x)}) \quad \varphi(0) \wedge \forall x(\varphi(x) \to \varphi(x+1)) \to \forall x\varphi(x),$$
$$(L_{\varphi(x)}) \quad \exists x\varphi(x) \to \exists x(\forall y < x\ \neg\varphi(y) \wedge \varphi(x)).$$

They are called the induction schema and the least number principle, respectively. *IOpen*, *LOpen* will denote the theory $PA^- \cup \{I_{\varphi(x)} \mid \varphi(x) : open\}$, $PA^- \cup \{L_{\varphi(x)} \mid \varphi(x) : open\}$, respectively.
In this paper we prove the equivalence of *IOpen* and *LOpen*. Van den Dries [v.d.D] noted that this can be proven model theoretically by using ideas in the proof of Shepherdson's theorem in [S1]. Our proof is syntactical and not model theoretical.

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§1. Introduction

The first order language $\mathcal{L}_A = \{0, 1, <, +, \cdot\}$, and the axioms of PA^- are the following:

1) $\forall x, y, z((x + y) + z = x + (y + z))$,
2) $\forall x, y(x + y = y + x)$,
3) $\forall x, y, z((x \cdot y) \cdot z = x \cdot (y \cdot z))$,
4) $\forall x, y(x \cdot y = y \cdot x)$,
5) $\forall x, y, z(x \cdot (y + z) = x \cdot y + x \cdot z)$,
6) $\forall x((x + 0 = x) \wedge (x \cdot 0 = 0))$,
7) $\forall x(x \cdot 1 = x)$,
8) $\forall x, y, z((x < y \wedge y < z) \to x < z)$,
9) $\forall x \neg(x < x)$,

10) $\forall x, y(x < y \vee x = y \vee y < x)$,
11) $\forall x, y, z(x < y \to x + z < y + z)$,
12) $\forall x, y, z((0 < z \wedge x < y) \to x \cdot z < y \cdot z)$,
13) $\forall x, y(x < y \to \exists z(x + z = y))$,
14) $0 < 1 \wedge \forall x(0 < x \to 1 \leq x)$,
15) $\forall x(0 \leq x)$.

We employ usual abbreviations such as $x \leq y < z$, x^n, $\forall x < y\ \varphi$ and so on. Note that 1)–13) imply converses to 11), 12) and that z in 13) is unique. We often represent the z by $y - x$.

Note that if we take Robinson Arithmetic Q as a 'base' theory, then *IOpen* and *LOpen* are equivalent to $Q \cup \{I_{\varphi(x)} \mid \varphi(x) : open\}$ and $Q \cup \{\forall x(x < x + 1)\} \cup \{L_{\varphi(x)} \mid \varphi(x) : open\}$, respectively (See [H-P], [K]).

It is easy to see that $PA^- \vdash L_{\neg\varphi(x)} \to I_{\varphi(x)}$ for any formula $\varphi(x)$, hence *LOpen* proves all axioms of *IOpen*. So we concentrate to see that *IOpen* proves the least number principle for open formulas.

Now we consider stronger systems than *IOpen*. Let $I\Sigma_n = PA^- \cup \{I_{\varphi(x)} \mid \varphi(x) : \Sigma_n\}$, which is equivalent to the theory $Q \cup \{I_{\varphi(x)} \mid \varphi(x) : \Sigma_n\}$. It is easy to see that $I\Sigma_n$ proves the least number principle for Σ_n-formulas. Because if we assume $\exists x\varphi(x) \wedge \forall x(\varphi(x) \to \exists y < x\ \varphi(y))$ and apply induction to the formula $\forall y < x\ \neg\varphi(y)$, then we obtain a contradiction. But we can not apply this argument for *IOpen*.

Shepherdson proved in [S1] that if $M \models PA^-$ and $R(M)$ is the real closure of the ordered ring corresponding to M, then $M \models IOpen$ iff $\forall r \in R(M)$ with $r > 0\ \exists s \in M$ such that $0 \leq r - s < 1$. One can easily prove that if the latter condition holds then M satisfies the least number principle for open formulas. Therefore we have that if $M \models IOpen$ then $M \models LOpen$, hence *IOpen* proves all the axioms of *LOpen*.

Our main result is Main Lemma 3.7 which states, roughly speaking, that "any open formula is equivalent to the union of finite intervals" is provable in *IOpen*. By using the Main Lemma, we will prove proof theoretically that *IOpen* proves the least number principle for open formulas. We begin investigation into atomic $\mathcal{L}_A$-formulas.

§2. Some properties of atomic formulas in PA^-

An atomic formula φ is of the form $s = t$ or of the form $s < t$, where s and t are terms. We will represent $\varphi(x)$ as an $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_0$ or $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 < b_n x^n + b_{n-1}x^{n-1} + \cdots + b_0$ where terms a_i and b_i do not contain the free variable x, and either $a_n \neq 0$ or $b_n \neq 0$. We say that $\varphi(x)$ is of degree n (in symbols: $deg_x\varphi = n$) if $\varphi(x)$ is represented as above. We often denote $deg_x\varphi$ simply by $deg\varphi$.

First we show in PA^- that if $\varphi(x)$ is an equality of degree n, then $\varphi(x)$ has at most n 'solutions'.

Proposition 2.1. *Let $\varphi(x)$ be an equality with $deg\varphi = n$ as above. Assume that $0 < n$. Then*

$$PA^- \vdash \bigvee_{i=0}^{n}(a_i \neq b_i) \to \neg\exists x_0, \cdots, x_n(x_0 < x_1 < \cdots < x_n \wedge \bigwedge_{i=0}^{n}\varphi(x_i)).$$

Proof. We construct proofs of the formula $\varphi(x)$ by induction on $n(=deg\varphi)$. For $n = 1$, we work in PA^-. Assume that there were x_0, x_1 such that $x_0 < x_1$, $\varphi(x_0)$ and $\varphi(x_1)$, that is,

$$\begin{aligned} a_1x_0 + a_0 &= b_1x_0 + b_0, \\ a_1x_1 + a_0 &= b_1x_1 + b_0. \end{aligned}$$

There exists the unique d such that $0 < d$ and $x_1 = x_0 + d$. Substituting $x_0 + d$ for x_1 in the second equation and using the first equation, we get $a_1d = b_1d$. Hence $a_1 = b_1$ since $0 < d$. It follows that $a_0 = b_0$, which is a contradiction. Therefore, this proposition holds in case $n = 1$.

Next, assume that the assertion holds for any equation with degree $n - 1$ and consider $\varphi(x)$ with $deg\varphi = n$. We work in PA^-. Suppose that there were $x_0, x_1, \cdots, x_n$ such that $x_0 < x_1 < \cdots < x_n$ and $\varphi(x_i)$ for each i, that is

$$(*)\left\{\begin{aligned} a_nx_0^n + a_{n-1}x_0^{n-1} + \cdots + a_0 &= b_nx_0^n + b_{n-1}x_0^{n-1} + \cdots + b_0, \\ a_nx_1^n + a_{n-1}x_1^{n-1} + \cdots + a_0 &= b_nx_1^n + b_{n-1}x_1^{n-1} + \cdots + b_0, \\ &\vdots \\ a_nx_n^n + a_{n-1}x_n^{n-1} + \cdots + a_0 &= b_nx_n^n + b_{n-1}x_n^{n-1} + \cdots + b_0. \end{aligned}\right.$$

There exists the unique d_i such that $x_{i+1} = x_0 + d_i$ for each $0 \le i \le n-1$, and $0 < d_0 < \cdots < d_{n-1}$. Let $\psi(y)$ be

$$a_n \sum_{k=1}^{n} \binom{n}{k} x_0^{n-k} y^{k-1} + a_{n-1} \sum_{k=1}^{n-1} \binom{n-1}{k} x_0^{n-k-1} y^{k-1} + \cdots + a_1$$
$$= b_n \sum_{k=1}^{n} \binom{n}{k} x_0^{n-k} y^{k-1} + b_{n-1} \sum_{k=1}^{n-1} \binom{n-1}{k} x_0^{n-k-1} y^{k-1} + \cdots + b_1.$$

Then $deg_y \psi = n-1$ and we have $\bigwedge_{i=0}^{n-1} \psi(d_i)$ by $(*)$. Let $\psi(y)$ represent $a'_{n-1} y^{n-1} + \cdots + a'_0 = b'_{n-1} y^{n-1} + \cdots + b'_0$ where a'_i, b'_i do not contain y. Then

$$\begin{array}{rcl} a'_{n-1} & = & a_n \binom{n}{n}, \\ a'_{n-2} & = & a_n \binom{n}{n-1} x_0 + a_{n-1} \binom{n-1}{n-1}, \\ a'_{n-3} & = & a_n \binom{n}{n-2} x_0^2 + a_{n-1} \binom{n-1}{n-2} x_0 + a_{n-2} \binom{n-2}{n-2}, \\ & \vdots & \\ a'_0 & = & a_n \binom{n}{1} x_0^{n-1} + a_{n-1} \binom{n-1}{1} x_0^{n-2} + \cdots + a_1 \binom{1}{1}. \end{array}$$

Similarly for b'_i. Thus, by the induction hypothesis it follows that $\bigwedge_{i=0}^{n-1} a'_i = b'_i$. Then $\bigwedge_{i=1}^{n} a_i = b_i$, hence $\bigwedge_{i=0}^{n} a_i = b_i$. Thus proposition holds for n. $\square$

For any formula $\varphi(x)$ we define inductively $\mathcal{J}_n(\varphi(x))$, which means that $\varphi(x)$ has at most n 'solutions'.

Definition 2.2. *For any formula $\varphi(x)$, we define $\mathcal{J}_n(\varphi(x))$ as follows inductively:*

$$\begin{array}{rcl} \mathcal{J}_0(\varphi(x)) & \equiv & \forall x \neg \varphi(x), \\ \mathcal{J}_{n+1}(\varphi(x)) & \equiv & \mathcal{J}_n(\varphi(x)) \vee \exists x_0, x_1, \cdots, x_n (x_0 < x_1 < \cdots < x_n \wedge \forall y (\varphi(y) \leftrightarrow \bigvee_{i=0}^{n} y = x_i)). \end{array}$$

Lemma 2.3. *Let $\varphi(x)$ be any formula. Then*

$$PA^- \vdash \neg \mathcal{J}_n(\varphi(x)) \rightarrow \exists x_0, x_1, \cdots, x_n (x_0 < x_1 < \cdots < x_n \wedge \bigwedge_{i=0}^{n} \varphi(x_i)).$$

Proof. Induction on n. We work in PA^-. The assertion is trivial for $n = 0$. For induction step, assume that the assertion holds for n. Suppose that $\neg\mathcal{J}_{n+1}(\varphi(x))$, that is $\neg\mathcal{J}_n(\varphi(x))$ and

$$\forall x_0, x_1, \cdots, x_n(x_0 < x_1 < \cdots < x_n \to \exists y((\varphi(y) \wedge \bigwedge_{i=0}^{n} y \neq x_i) \vee (\neg\varphi(y) \wedge \bigvee_{i=0}^{n} y = x_i))).$$

By $\neg\mathcal{J}_n(\varphi(x))$ and the induction hypothesis, we have $a_0, a_1, \cdots, a_n$ such that $a_0 < a_1 < \cdots < a_n$ and $\bigwedge_{i=0}^{n} \varphi(a_i)$. Then there exists a b such that

$$(\varphi(b) \wedge \bigwedge_{i=0}^{n} b \neq a_i) \vee (\neg\varphi(b) \wedge \bigvee_{i=0}^{n} b = a_i).$$

Since $\bigwedge_{i=0}^{n} \varphi(a_i)$, $\varphi(b) \wedge \bigwedge_{i=0}^{n} b \neq a_i$ holds. It is easy to see that

$$\begin{aligned} a_0 < a_1 < \cdots < a_n \wedge \bigwedge_{i=0}^{n} b \neq a_i \to \quad & (b < a_0 < a_1 < \cdots < a_n \vee \\ & a_0 < b < a_1 < \cdots < a_n \vee \\ & \cdots \\ & a_0 < a_1 < \cdots < b < a_n \vee \\ & a_0 < a_1 < \cdots < a_n < b), \end{aligned}$$

then we have $\exists x_0, x_1, \cdots, x_{n+1}(x_0 < x_1 < \cdots < x_{n+1} \wedge \bigwedge_{i=0}^{n+1} \varphi(x_i))$. □

Corollary 2.4. *Let $\varphi(x)$ be an equality with $deg\varphi = n$ and $0 < n$. Then*

$$PA^- \vdash \bigvee_{i=0}^{n} (a_i \neq b_i) \to \mathcal{J}_n(\varphi(x))$$

Proof. Immediate from Proposition 2.1 and Lemma 2.3. □

Let $\varphi(x)$ be an inequality. Consider a sequence $x_0 < x_1 < \cdots < x_k$ with $\varphi(x_0) \wedge \neg\varphi(x_1) \wedge \cdots \wedge \overbrace{\neg \cdots \neg}^{i \text{ times}} \varphi(x_i) \wedge \cdots \wedge \overbrace{\neg \cdots \neg}^{k \text{ times}} \varphi(x_k)$ or $\neg\varphi(x_0) \wedge \neg\neg\varphi(x_1) \wedge \cdots \wedge \overbrace{\neg \cdots \neg}^{i+1 \text{ times}} \varphi(x_i) \wedge \cdots \wedge \overbrace{\neg \cdots \neg}^{k+1 \text{ times}} \varphi(x_k)$, in which affirmation alternates with negation since $\vdash \neg\neg\psi \leftrightarrow \psi$ for any formula ψ. We shall show that the number of such alternation is bounded by $deg_x\varphi$. For this purpose we introduce the following notation.

Definition 2.5. *For terms $t_0, t_1, \cdots, t_k$, and $i + j \leq k$,*

$$\mathcal{S}^n_{i,j}(t_0, t_1, \cdots, t_k) = \sum_{n_0+n_1+\cdots+n_j=n} t_i^{n_0} \cdot t_{i+1}^{n_1} \cdot \;\cdots\; \cdot t_{i+j}^{n_j}.$$

For example, $\mathcal{S}^n_{0,0}(x, y) = x^n$, $\mathcal{S}^n_{0,1}(x, y, z) = x^n + x^{n-1}y + \cdots + y^n$, $\mathcal{S}^n_{1,1}(x, y, z) = y^n + y^{n-1}z + \cdots + z^n$, and so on. Especially $\mathcal{S}^0_{i,j}(t_0, t_1, \cdots, t_k) = 1$.

Lemma 2.6. *Let $t_0, t_1, \cdots, t_k$ be terms and $i + j + 1 \leq k$. Then*

$$\begin{aligned} PA^- \vdash t_0 < t_1 < \cdots < t_k \to \;& \mathcal{S}^n_{i,j}(t_0, t_1, \cdots, t_k) < \mathcal{S}^n_{i+1,j}(t_0, t_1, \cdots, t_k), \\ PA^- \vdash t_0 < t_1 < \cdots < t_k \to \;& \mathcal{S}^n_{i+1,j}(t_0, t_1, \cdots, t_k) - \mathcal{S}^n_{i,j}(t_0, t_1, \cdots, t_k) \\ & = (t_{i+j+1} - t_i) \cdot \mathcal{S}^{n-1}_{i,j+1}(t_0, t_1, \cdots, t_k). \end{aligned}$$

$$\left(\begin{aligned} &\text{More precisely,} \\ &PA^- \vdash t_0 < t_1 < \cdots < t_k \to \;\; \exists d(t_{i+j+1} = t_i + d \wedge \mathcal{S}^n_{i+1,j}(t_0, \cdots, t_k) \\ &\qquad\qquad\qquad\qquad\qquad = \mathcal{S}^n_{i,j}(t_0, t_1, \cdots, t_k) + d \cdot \mathcal{S}^{n-1}_{i,j+1}(t_0, t_1, \cdots, t_k)). \end{aligned} \right)$$

Proof. The first assertion is clear. The second assertion is proved in PA^- by easy computation. We work in PA^-. Assume $t_0 < t_1 < \cdots < t_k$.

$$\begin{aligned} &\mathcal{S}^n_{i+1,j}(t_0, \cdots, t_k) - \mathcal{S}^n_{i,j}(t_0, \cdots, t_k) \\ &= \sum_{n_0+\cdots+n_j=n} t_{i+1}^{n_0} \cdots t_{i+j+1}^{n_j} - \sum_{n_0+\cdots+n_j=n} t_i^{n_0} \cdots t_{i+j}^{n_j} \\ &= \sum_{n_0+\cdots+n_j=n} t_{i+1}^{n_0} \cdots t_{i+j}^{n_{j}-1} t_{i+j+1}^{n_j} - \sum_{n_0+\cdots+n_j=n} t_{i+1}^{n_0} \cdots t_{i+j}^{n_{j-1}} t_i^{n_j} \\ &= \sum_{n_0+\cdots+n_j=n} t_{i+1}^{n_0} \cdots t_{i+j}^{n_{j-1}} (t_{i+j+1}^{n_j} - t_i^{n_j}) \\ &= (t_{i+j+1} - t_i) \sum_{n_0+\cdots+n_j=n} t_{i+1}^{n_0} \cdots t_{i+j}^{n_{j-1}} \sum_{m_0+m_1=n_j-1} t_{i+j+1}^{m_0} t_i^{m_1} \\ &= (t_{i+j+1} - t_i) \sum_{n_0+\cdots+n_j+m_0+m_1=n-1} t_{i+1}^{n_0} \cdots t_{i+j}^{n_{j-1}} t_{i+j+1}^{m_0} t_i^{m_1} \\ &= (t_{i+j+1} - t_i) \sum_{n_0+\cdots+n_{j+1}=n-1} t_i^{n_0} \cdots t_{i+j+1}^{n_{j+1}} \\ &= (t_{i+j+1} - t_i) \cdot \mathcal{S}^{n-1}_{i,j+1}(t_0, \cdots, t_k). \end{aligned}$$

□

Proposition 2.7. *Let $\varphi(x)$ be an inequality with $deg\varphi = n$ and $0 < n$. Then*

$$\begin{aligned} PA^- \vdash \ & \neg\exists x_0, \cdots, x_{n+1}(x_0 < x_1 < \cdots < x_{n+1} \wedge \\ & ((\varphi(x_0) \wedge \neg\varphi(x_1) \wedge \cdots \wedge \overbrace{\neg\cdots\neg}^{n+1 \text{ times}} \varphi(x_{n+1})) \vee \\ & (\neg\varphi(x_0) \wedge \neg\neg\varphi(x_1) \wedge \cdots \wedge \overbrace{\neg\cdots\neg}^{n+2 \text{ times}} \varphi(x_{n+1})))). \end{aligned}$$

Proof. We work in PA^-. Assume that $x_0 < x_1 < \cdots < x_{n+1}$, and $\varphi(x_0) \wedge \neg\varphi(x_1) \wedge \cdots \wedge \overbrace{\neg\cdots\neg}^{n+1 \text{ times}} \varphi(x_{n+1})$, that is,

$$\Lambda_0 \begin{cases} a_n\mathcal{S}^n_{0,0}(\vec{x}) + a_{n-1}\mathcal{S}^{n-1}_{0,0}(\vec{x}) + \cdots + a_0 & < \quad b_n\mathcal{S}^n_{0,0}(\vec{x}) + b_{n-1}\mathcal{S}^{n-1}_{0,0}(\vec{x}) + \cdots + b_0, \\ a_n\mathcal{S}^n_{1,0}(\vec{x}) + a_{n-1}\mathcal{S}^{n-1}_{1,0}(\vec{x}) + \cdots + a_0 & \geq \quad b_n\mathcal{S}^n_{1,0}(\vec{x}) + b_{n-1}\mathcal{S}^{n-1}_{1,0}(\vec{x}) + \cdots + b_0, \\ a_n\mathcal{S}^n_{2,0}(\vec{x}) + a_{n-1}\mathcal{S}^{n-1}_{2,0}(\vec{x}) + \cdots + a_0 & < \quad b_n\mathcal{S}^n_{2,0}(\vec{x}) + b_{n-1}\mathcal{S}^{n-1}_{2,0}(\vec{x}) + \cdots + b_0, \\ & \vdots \\ a_n\mathcal{S}^n_{n+1,0}(\vec{x}) + a_{n-1}\mathcal{S}^{n-1}_{n+1,0}(\vec{x}) + \cdots + a_0 & \diamond \quad b_n\mathcal{S}^n_{n+1,0}(\vec{x}) + b_{n-1}\mathcal{S}^{n-1}_{n+1,0}(\vec{x}) + \cdots + b_0, \end{cases}$$

where '$\vec{x}$' is '$x_0, \cdots, x_{n+1}$' and '$\diamond$' is '$<$' if $n+1$ is even, and is '$\geq$' if $n+1$ is odd. The system Λ_0 consists of $n+2$ inequalities. By the first and the second inequalities, we have

$$\begin{aligned} & a_n(\mathcal{S}^n_{1,0}(\vec{x}) - \mathcal{S}^n_{0,0}(\vec{x})) + a_{n-1}(\mathcal{S}^{n-1}_{1,0}(\vec{x}) - \mathcal{S}^{n-1}_{0,0}(\vec{x})) + \cdots + a_1(\mathcal{S}^1_{1,0}(\vec{x}) - \mathcal{S}^1_{0,0}(\vec{x})) \\ & \quad > b_n(\mathcal{S}^n_{1,0}(\vec{x}) - \mathcal{S}^n_{0,0}(\vec{x})) + b_{n-1}(\mathcal{S}^{n-1}_{1,0}(\vec{x}) - \mathcal{S}^{n-1}_{0,0}(\vec{x})) + \cdots + b_1(\mathcal{S}^1_{1,0}(\vec{x}) - \mathcal{S}^1_{0,0}(\vec{x})), \end{aligned}$$

hence by using Lemma 2.6,

$$a_n\mathcal{S}^{n-1}_{0,1}(\vec{x}) + a_{n-1}\mathcal{S}^{n-2}_{0,1}(\vec{x}) + \cdots + a_1 > b_n\mathcal{S}^{n-1}_{0,1}(\vec{x}) + b_{n-1}\mathcal{S}^{n-2}_{0,1}(\vec{x}) + \cdots + b_1.$$

In the same way, we have

$$\Lambda_1 \begin{cases} a_n\mathcal{S}^{n-1}_{0,1}(\vec{x}) + a_{n-1}\mathcal{S}^{n-2}_{0,1}(\vec{x}) + \cdots + a_1 & > \quad b_n\mathcal{S}^{n-1}_{0,1}(\vec{x}) + b_{n-1}\mathcal{S}^{n-2}_{0,1}(\vec{x}) + \cdots + b_1, \\ a_n\mathcal{S}^{n-1}_{1,1}(\vec{x}) + a_{n-1}\mathcal{S}^{n-2}_{1,1}(\vec{x}) + \cdots + a_1 & < \quad b_n\mathcal{S}^{n-1}_{1,1}(\vec{x}) + b_{n-1}\mathcal{S}^{n-2}_{1,1}(\vec{x}) + \cdots + b_1, \\ a_n\mathcal{S}^{n-1}_{2,1}(\vec{x}) + a_{n-1}\mathcal{S}^{n-2}_{2,1}(\vec{x}) + \cdots + a_1 & > \quad b_n\mathcal{S}^{n-1}_{2,1}(\vec{x}) + b_{n-1}\mathcal{S}^{n-2}_{2,1}(\vec{x}) + \cdots + b_1, \\ & \vdots \\ a_n\mathcal{S}^{n-1}_{n,1}(\vec{x}) + a_{n-1}\mathcal{S}^{n-2}_{n,1}(\vec{x}) + \cdots + a_1 & \diamond' \quad b_n\mathcal{S}^{n-1}_{n,1}(\vec{x}) + b_{n-1}\mathcal{S}^{n-2}_{n,1}(\vec{x}) + \cdots + b_1, \end{cases}$$

where '$\diamond'$' is '$<$' if $n+1$ is even and is '$>$' if $n+1$ is odd. The system Λ_1 consists of $n+1$ inequalities. We continue further this reduction $n-2$ times. Then we get that, in the case that n is even

$$\Lambda_{n-1} \begin{cases} a_n\mathcal{S}^1_{0,n-1}(\vec{x}) + a_{n-1} > b_n\mathcal{S}^1_{0,n-1}(\vec{x}) + b_{n-1}, \\ a_n\mathcal{S}^1_{1,n-1}(\vec{x}) + a_{n-1} < b_n\mathcal{S}^1_{1,n-1}(\vec{x}) + b_{n-1}, \\ a_n\mathcal{S}^1_{2,n-1}(\vec{x}) + a_{n-1} > b_n\mathcal{S}^1_{2,n-1}(\vec{x}) + b_{n-1}, \end{cases}$$

and in the case that n is odd,

$$\Lambda'_{n-1}\left\{\begin{array}{l} a_n\mathcal{S}^1_{0,n-1}(\vec{x})+a_{n-1}<b_n\mathcal{S}^1_{0,n-1}(\vec{x})+b_{n-1},\\ a_n\mathcal{S}^1_{1,n-1}(\vec{x})+a_{n-1}>b_n\mathcal{S}^1_{1,n-1}(\vec{x})+b_{n-1},\\ a_n\mathcal{S}^1_{2,n-1}(\vec{x})+a_{n-1}<b_n\mathcal{S}^1_{2,n-1}(\vec{x})+b_{n-1}.\end{array}\right.$$

We proceed the same reduction once again. We have

$$\Lambda_n\left\{\begin{array}{l} a_n\mathcal{S}^0_{0,n}(\vec{x})<b_n\mathcal{S}^0_{0,n}(\vec{x}),\\ a_n\mathcal{S}^0_{1,n}(\vec{x})>b_n\mathcal{S}^0_{1,n}(\vec{x}),\end{array}\right. \quad\text{or}\quad \Lambda'_n\left\{\begin{array}{l} a_n\mathcal{S}^0_{0,n}(\vec{x})>b_n\mathcal{S}^0_{0,n}(\vec{x}),\\ a_n\mathcal{S}^0_{1,n}(\vec{x})<b_n\mathcal{S}^0_{1,n}(\vec{x}).\end{array}\right.$$

In any case, we have $a_n < b_n \wedge b_n < a_n$, which is a contradiction.

Since we get a contradiction in the same way when we assume that $x_0 < x_1 < \cdots < x_{n+1}$ and $\neg\varphi(x_0)\wedge\neg\neg\varphi(x_1)\wedge\cdots \overbrace{\neg\cdots\neg}^{n+2 \text{ times}} \varphi(x_{n+1})$, we have proved the proposition. □

§3. *IOpen* is equivalent to *LOpen*

Let $\varphi(x)$ be an open formula. Suppose $\varphi(x_0)$ and $\neg\varphi(x_1)$ with $x_0 < x_1$. Intuitively it is clear that we can find a x such that $\varphi(x)$ and $\neg\varphi(x+1)$ with $x_0 \le x < x_1$. In fact this can be proved in *IOpen*.

Lemma 3.1. *Let $\varphi(x)$ be an open formula. Then*

$$IOpen \vdash \forall x_0, x_1(x_0 < x_1 \wedge \varphi(x_0) \wedge \neg\varphi(x_1) \to \exists x(x_0 \le x < x_1 \wedge \varphi(x) \wedge \neg\varphi(x+1))).$$

Proof. Assume that $x_0 < x_1 \wedge \varphi(x_0) \wedge \neg\varphi(x_1)$. There exists the unique d such that $x_1 = x_0 + d$ since $x_0 < x_1$. We represent the d by $x_1 - x_0$ and $\psi(z)$ as

$$\varphi(x_0 + z) \wedge 0 \le z \le x_1 - x_0.$$

Then $\psi(0)$ and $\neg\psi(x_1 - x_0)$, hence $\psi(0) \wedge \exists x\neg\psi(x)$. By using $I_{\psi(z)}$ we have $\exists z(\psi(z)\wedge\neg\psi(z+1))$, consequently $\exists x(\varphi(x)\wedge\neg\varphi(x+1)\wedge x_0 \le x \le x_1)$. Finaly $x \neq x_1$ follows from $\neg\varphi(x_1)$. □

Note that the $\mathcal{L}_A$-sentence in Lemma 3.1 implies $I_{\varphi(x)}$ in PA^-. Let us say that a formula of the form $x < a$, $a \le x < b$, or $b \le x$ is an interval. Let $\varphi(x)$ be an inequality of degree n. We show in *IOpen* that there exist $x_0, x_1, \cdots, x_{n+1}$ such that $\varphi(x)$ is equivalent to the disjunction of intervals whose endpoints are $x_0, x_1, \cdots, x_{n+1}$. Note that the number of these intervals is at most $\lfloor\frac{n}{2}\rfloor + 1$ since it may be the case that $x_{i+1} = x_i$ for some i. To be precise, we define inductively the formula $\mathcal{I}_n(\varphi(x))$ as follows.

Definition 3.2. *For any formula $\varphi(x)$, we define $\mathcal{I}_n(\varphi(x))$ as follows inductively:*

$$\begin{aligned}\mathcal{I}_0(\varphi(x)) &\equiv \forall x\varphi(x) \vee \forall x\neg\varphi(x),\\ \mathcal{I}_{n+1}(\varphi(x)) &\equiv \mathcal{I}_n(\varphi(x)) \vee \theta_{n+1}(\varphi(x)) \vee \theta'_{n+1}(\varphi(x)),\end{aligned}$$

where

$$\begin{aligned}\theta_{2m+1}(\varphi(x)) \equiv\ & \exists x_0, x_1, \cdots, x_{2m}(0 < x_0 < x_1 < \cdots < x_{2m} \wedge \\ & (\forall y(\varphi(y) \leftrightarrow y < x_0 \vee (\bigvee_{i=1}^{m} x_{2i-1} \le y < x_{2i})))),\\ \theta'_{2m+1}(\varphi(x)) \equiv\ & \exists x_0, x_1, \cdots, x_{2m}(0 < x_0 < x_1 < \cdots < x_{2m} \wedge \\ & (\forall y(\varphi(y) \leftrightarrow (\bigvee_{i=0}^{m-1} x_{2i} \le y < x_{2i+1}) \vee x_{2m} \le y))),\\ \theta_{2m+2}(\varphi(x)) \equiv\ & \exists x_0, x_1, \cdots, x_{2m+1}(0 < x_0 < x_1 < \cdots < x_{2m+1} \wedge \\ & (\forall y(\varphi(y) \leftrightarrow y < x_0 \vee (\bigvee_{i=1}^{m} x_{2i-1} \le y < x_{2i}) \vee x_{2m+1} \le y))),\\ \theta'_{2m+2}(\varphi(x)) \equiv\ & \exists x_0, x_1, \cdots, x_{2m+1}(0 < x_0 < x_1 < \cdots < x_{2m+1} \wedge \\ & (\forall y(\varphi(y) \leftrightarrow \bigvee_{i=0}^{m} x_{2i} \le y < x_{2i+1}))).\end{aligned}$$

Note that $\vdash \mathcal{I}_n(\varphi(x)) \to \mathcal{I}_{n+1}(\varphi(x))$ for any formula $\varphi(x)$ and any n, and

$$\begin{aligned}PA^- \vdash\ & 0 < x_0 < x_1 < \cdots < x_{2m} \to \\ & (\neg(y < x_0 \vee (\bigvee_{i=1}^{m} x_{2i-1} \le y < x_{2i})) \leftrightarrow (\bigvee_{i=0}^{m-1} x_{2i} \le y < x_{2i+1}) \vee x_{2m} \le y),\\ PA^- \vdash\ & 0 < x_0 < x_1 < \cdots < x_{2m+1} \to \\ & (\neg(y < x_0 \vee (\bigvee_{i=1}^{m} x_{2i-1} \le y < x_{2i}) \vee x_{2m+1} \le y) \leftrightarrow \bigvee_{i=0}^{m} x_{2i} \le y < x_{2i+1}).\end{aligned}$$

Lemma 3.3. *Let $\varphi(x)$ be an open formula. Then*

$$\begin{aligned}&1)\ \ IOpen \vdash \ \ \varphi(0) \wedge \neg\mathcal{I}_n(\varphi(x)) \to \exists x_0, x_1, \cdots, x_{n+1}(x_0 < x_1 < \cdots < x_{n+1} \wedge \\ &\qquad\qquad \varphi(x_0) \wedge \neg\varphi(x_1) \wedge \cdots \wedge \overbrace{\neg\cdots\neg}^{n+1 \text{ times}} \varphi(x_{n+1})),\\ &2)\ \ IOpen \vdash \neg\varphi(0) \wedge \neg\mathcal{I}_n(\varphi(x)) \to \exists x_0, x_1, \cdots, x_{n+1}(x_0 < x_1 < \cdots < x_{n+1} \wedge \\ &\qquad\qquad \neg\varphi(x_0) \wedge \neg\neg\varphi(x_1) \wedge \cdots \wedge \overbrace{\neg\cdots\neg}^{n+2 \text{ times}} \varphi(x_{n+1})).\end{aligned}$$

Proof. 1). By induction on n. Trivial for $n = 0$; thus assume that the assertion holds for $n = 2m$ and we work in *IOpen*. Suppose $\varphi(0) \wedge \neg\mathcal{I}_{2m+1}(\varphi(x))$, that is, $\varphi(0) \wedge \neg\mathcal{I}_{2m}(\varphi(x)) \wedge \neg\theta_{2m+1}(\varphi(x)) \wedge \neg\theta'_{2m+1}(\varphi(x))$. By the induction hypothesis, $\varphi(0)$ and $\neg\mathcal{I}_{2m}(\varphi(x))$, there exist $a_0, a_1, \cdots, a_{2m+1}$ such that

$a_0 < a_1 < \cdots < a_{2m+1}$ and $\varphi(a_0), \neg\varphi(a_1), \cdots, \overbrace{\neg\cdots\neg}^{2m+1\ \text{times}} \varphi(a_{2m+1})$. By Lemma 3.1 there exist $b_0, b_1, \cdots, b_{2m}$ such that $a_0 \leq b_0 < a_1 \leq b_1 < a_2 \leq \cdots < a_{2m} \leq b_{2m} < a_{2m+1}$ and

$$\begin{array}{rcr} \varphi(b_0) & , & \neg\varphi(b_0+1) \\ \neg\varphi(b_1) & , & \neg\neg\varphi(b_1+1) \\ & \vdots & \\ \overbrace{\neg\cdots\neg}^{2m\ \text{times}} \varphi(b_{2m}) & , & \overbrace{\neg\cdots\neg}^{2m+1\ \text{times}} \varphi(b_{2m}+1). \end{array}$$

Since $0 < b_0+1 < b_1+1 < \cdots < b_{2m}+1$, we apply $b_0+1, b_1+1, \cdots, b_{2m}+1$ to $\neg\theta_{2m+1}(\varphi(x))$. Then we have

$$\exists y(\varphi(y) \wedge ((\bigvee_{i=0}^{m-1} b_{2i}+1 \leq y < b_{2i+1}+1) \vee b_{2m}+1 \leq y)) \quad \text{or}$$

$$\exists y(\neg\varphi(y) \wedge (y < b_0+1 \vee (\bigvee_{i=1}^{m} b_{2i-1}+1 \leq y < b_{2i}+1))).$$

Case 1.

Let $b_{2i}+1 \leq c < b_{2i+1}+1$ and $\varphi(c)$ for some i with $0 \leq i < m$. Since $\neg\varphi(b_{2i}+1)$ and $\neg\varphi(b_{2i+1})$, we have $b_{2i}+1 < c < b_{2i+1}$. Then we find $2m+3$ terms, that is, $b_0 < \cdots < b_{2i} < b_{2i}+1 < c < b_{2i+1} < \cdots < b_{2m}$ such that

$$\varphi(b_0), \ldots, \varphi(b_{2i}), \neg\varphi(b_{2i}+1), \varphi(c), \neg\varphi(b_{2i+1}), \cdots, \varphi(b_{2m}).$$

Let $b_{2m}+1 \leq c$ and $\varphi(c)$. Since $\neg\varphi(b_{2m}+1)$, we find $2m+3$ terms, that is, $b_0 < b_1 < \cdots < b_{2m} < b_{2m}+1 < c$ such that

$$\varphi(b_0), \neg\varphi(b_1), \cdots, \varphi(b_{2m}), \neg\varphi(b_{2m}+1), \varphi(c).$$

Case 2.

Let $c < b_0+1$ and $\neg\varphi(c)$. Since $\varphi(0)$ and $\varphi(b_0)$, we have $0 < c < b_0$. Then we find $2m+3$ terms, that is, $0 < c < b_0 < b_1 < \cdots < b_{2m}$ such that

$$\varphi(0), \neg\varphi(c), \varphi(b_0), \cdots, \varphi(b_{2m}).$$

Let $b_{2i-1}+1 \leq c < b_{2i}+1$ and $\neg\varphi(c)$ for some i with $1 \leq c \leq m$. Since $\varphi(b_{2i-1}+1)$ and $\varphi(b_{2i})$, we have $b_{2i-1}+1 < c < b_{2i}$. Then we find $2m+3$ terms, that is, $b_0 < \cdots < b_{2i-1} < b_{2i-1}+1 < c < b_{2i} < \cdots < b_{2m}$ such that

$$\varphi(b_0), \cdots, \neg\varphi(b_{2i-1}), \varphi(b_{2i-1}+1), \neg\varphi(c), \varphi(b_{2i}), \cdots, \varphi(b_{2m}).$$

Thus we have proved 1) for $n = 2m+1$. We can prove similarly that 1) for $n = 2m+1$ implies 1) for $n = 2m+2$.

2). It is easy to see that $PA^- \vdash \mathcal{I}_n(\neg\psi(x)) \leftrightarrow \mathcal{I}_n(\psi(x))$ for any formula $\psi(x)$ by induction on n. Then we have the assertion 2) by 1). $\Box$

Corollary 3.4. *Let $\varphi(x)$ be an inequality with $deg\varphi = n$ and $0 < n$. Then*

$$IOpen \vdash \mathcal{I}_n(\varphi(x)).$$

Proof. Immediate from Proposition 2.7 and Lemma 3.3. □

In order to deal with an arbitrary open formula, we need some technical lemmas. Now recall Corollary 2.4, which says that any equality $\varphi(x)$ with $deg_x\varphi = n$ has at most n 'solutions'. We rewrite Corollary 2.4 in terms of $\mathcal{I}_n(\varphi(x))$.

Lemma 3.5. *Let $\varphi(x)$ be an equality with $deg\varphi = n$ and $0 < n$. Then*

$$PA^- \vdash \mathcal{I}_{2n+2}(\varphi(x)).$$

Proof. We work in PA^-. If $a_i = b_i$ for all i with $0 \leq i \leq n$, we have $\forall x\varphi(x)$, then $\mathcal{I}_0(\varphi(x))$. By the definition for $\mathcal{I}$, we have $\mathcal{I}_{2n+2}(\varphi(x))$. Otherwise, since $y = a \leftrightarrow a \leq y < a+1$ we have the assertion by Corollary 2.4. □

Next we show that the set of formulas $\varphi(x)$ for which $\mathcal{I}_n(\varphi(x))$ are provable in PA^- for some n is closed under conjunction and disjunction.

Lemma 3.6. *For any formulas $\varphi(x)$ and $\psi(x)$*

$$PA^- \vdash \mathcal{I}_n(\varphi(x)) \wedge \mathcal{I}_m(\psi(x)) \to \mathcal{I}_{n+m}(\varphi(x) \wedge \psi(x)),$$
$$PA^- \vdash \mathcal{I}_n(\varphi(x)) \wedge \mathcal{I}_m(\psi(x)) \to \mathcal{I}_{n+m}(\varphi(x) \vee \psi(x)).$$

Proof. We first show the following results: For $0 < n$ and $0 < m$

$$PA^- \vdash \Theta_n(\varphi(x)) \wedge \Theta_m(\psi(x)) \to \mathcal{I}_{n+m}(\varphi(x) \wedge \psi(x)),$$
$$PA^- \vdash \Theta_n(\varphi(x)) \wedge \Theta_m(\psi(x)) \to \mathcal{I}_{n+m}(\varphi(x) \vee \psi(x)),$$

where Θ_k is θ_k or θ'_k. We work in PA^-. Assume that $\theta_{2n+1}(\varphi(x))$ and $\theta'_{2m+2}(\psi(x))$. There exist $a_0, a_1, \cdots, a_{2n}$, and $b_0, b_1, \cdots, b_{2m+1}$ such that $0 < a_0 < a_1 < \cdots < a_{2n}$ and $0 < b_0 < b_1 < \cdots < b_{2m+1}$ and

$$\forall y(\varphi(y) \quad \leftrightarrow \quad y < a_0 \vee (\bigvee_{i=1}^{n} a_{2i-1} \leq y < a_{2i})),$$
$$\forall y(\varphi(y) \quad \leftrightarrow \quad \bigvee_{i=0}^{m} b_{2i} \leq y < b_{2i+1}).$$

Consider all the possibilities of ordering between $a_0, a_1, \cdots, a_{2n}$ and $b_0, b_1, \cdots, b_{2m+1}$. For example; if $2n \leq 2m+1$ and $a_0 < b_0 < b_1 < a_1 < \cdots < a_{2i} < b_{2i} < b_{2i+1} < a_{2i+1} < \cdots < a_{2n} < b_{2n} < b_{2n+1} < \cdots < b_{2m+1}$ then we have

$$\begin{aligned}
&\forall x \neg(\varphi(x) \wedge \psi(x)),\\
&\forall y((\varphi(y) \vee \psi(y)) \leftrightarrow (y < a_0 \vee (\bigvee_{i=1}^{n} a_{2i-1} \leq y < a_{2i}) \vee (\bigvee_{i=0}^{m} b_{2i} \leq y < b_{2i+1})),
\end{aligned}$$

thus $\mathcal{I}_0(\varphi(x) \wedge \psi(x))$ and $\mathcal{I}_{2n+1+2m+2}(\varphi(x) \vee \psi(x))$. If $2m+1 < 2n$ and $b_0 < a_0 < a_1 < b_1 < \cdots < b_{2i} < a_{2i} < a_{2i+1} < b_{2i+1} < \cdots < a_{2m+1} < b_{2m+1} < a_{2m+2} < a_{2m+3} < \cdots < a_{2n}$ then we have

$$\begin{aligned}
&\forall y((\varphi(y) \wedge \psi(y)) \leftrightarrow ((\bigvee_{i=0}^{m} b_{2i} \leq y < a_{2i}) \vee (\bigvee_{i=0}^{m} a_{2i+1} \leq y < b_{2i+1}))),\\
&\forall y((\varphi(y) \vee \psi(y)) \leftrightarrow (y < a_{2m+2} \vee (\bigvee_{i=m+2}^{n} a_{2i-1} \leq y < a_{2i}))),
\end{aligned}$$

thus $\mathcal{I}_{4m+4}(\varphi(x) \wedge \psi(x))$ and $\mathcal{I}_{2n-2m-1}(\varphi(x) \vee \psi(x))$. Similarly, for the other orderings we have $\mathcal{I}_k(\varphi(x) \wedge \psi(x))$ and $\mathcal{I}_l(\varphi(x) \vee \psi(x))$ for some $k, l \leq 2n+1+2m+2$. So we have $\mathcal{I}_{2n+1+2m+2}(\varphi(x) \wedge \psi(x))$ and $\mathcal{I}_{2n+1+2m+2}(\varphi(x) \vee \psi(x))$. Therefore we get the result in this case. Similarly for the other cases on Θ_n and Θ_m, and hence we can show the first results.

Secondly we show the following results: For $0 < m$

$$\begin{aligned}
PA^- &\vdash \mathcal{I}_n(\varphi(x)) \wedge \Theta_m(\psi(x)) \rightarrow \mathcal{I}_{n+m}(\varphi(x) \wedge \psi(x)),\\
PA^- &\vdash \mathcal{I}_n(\varphi(x)) \wedge \Theta_m(\psi(x)) \rightarrow \mathcal{I}_{n+m}(\varphi(x) \vee \psi(x)),
\end{aligned}$$

where Θ_m is θ_m or θ'_m. We prove these results by induction on n. Let $n = 0$, and we work in PA^-. Recall that $\mathcal{I}_0(\varphi(x))$ is $\forall x \varphi(x) \vee \forall x \neg\varphi(x)$. If $\forall x \varphi(x)$, we have $\forall x(\varphi(x) \wedge \psi(x) \leftrightarrow \psi(x))$ and $\forall x(\varphi(x) \vee \psi(x))$. Then $\Theta_m(\psi(x))$ implies $\mathcal{I}_m(\varphi(x) \wedge \psi(x))$ and $\mathcal{I}_0(\varphi(x) \vee \psi(x))$. If $\forall x \neg\varphi(x)$, we have $\forall x \neg(\varphi(x) \wedge \psi(x))$ and $\forall x(\varphi(x) \vee \psi(x) \leftrightarrow \psi(x))$. Then $\Theta_m(\psi(x))$ implies $\mathcal{I}_0(\varphi(x) \wedge \psi(x))$ and $\mathcal{I}_m(\varphi(x) \vee \psi(x))$. In both cases, we have the second results for $n = 0$. Assume the results for n and $\mathcal{I}_{n+1}(\varphi(x)) \wedge \Theta_m(\psi(x))$, that is, $(\mathcal{I}_n(\varphi(x)) \wedge \Theta_m(\psi(x))) \vee (\theta_{n+1}(\varphi(x)) \wedge \Theta_m(\psi(x))) \vee (\theta'_{n+1}(\varphi(x)) \wedge \Theta_m(\psi(x)))$. If $\mathcal{I}_n(\varphi(x)) \wedge \Theta_m(\psi(x))$, we have $\mathcal{I}_{n+m}(\varphi(x) \wedge \psi(x))$ and $\mathcal{I}_{n+m}(\varphi(x) \vee \psi(x))$ by the induction hypothesis. If $\theta_{n+1}(\varphi(x)) \wedge \Theta_m(\psi(x))$ or $\theta'_{n+1}(\varphi(x)) \wedge \Theta_m(\psi(x))$, we have $\mathcal{I}_{n+1+m}(\varphi(x) \wedge \psi(x))$ and $\mathcal{I}_{n+1+m}(\varphi(x) \vee \psi(x))$ by the first result. Thus the second results are proved.

Finally the assertion in this lemma is proved by induction on m, by using the second results. $\square$

Now let $\varphi(x)$ be an open formula. Let a disjunctive normal form of $\varphi(x)$ be $\bigvee_{j=0}^{q}\bigwedge_{i=0}^{p_j}\varphi_{ij}(x)$, where $\varphi_{ij}(x)$ is atomic or negated atomic. Since $\neg s = t$ and $\neg s < t$ are equivalent to $s < t \vee t < s$ and $t < s \vee t = s$ in PA^- respectively, we may suppose, by using the distributive law, $\varphi(x) \equiv \bigvee_{j=0}^{q}\bigwedge_{i=0}^{p_j}\varphi_{ij}(x)$ where $\varphi_{ij}(x)$ is of the form $s < t$ or $s = t$.

Main Lemma 3.7. *For any open formula $\varphi(x)$ there exists an n such that*

$$IOpen \vdash \mathcal{I}_n(\varphi(x)),$$

that is,

1) *for any atomic formula $\varphi(x)$ there exists an n such that* *$IOpen \vdash \mathcal{I}_n(\varphi(x))$,*
2) *$IOpen \vdash \mathcal{I}_n(\varphi(x))$ and $IOpen \vdash \mathcal{I}_m(\psi(x))$ then,*
   2.1) *$IOpen \vdash \mathcal{I}_{n+m}(\varphi(x) \wedge \psi(x))$,*
   2.2) *$IOpen \vdash \mathcal{I}_{n+m}(\varphi(x) \vee \psi(x))$.*

Proof. 1) is immediately obtained from Corollary 3.4 and Lemma 3.5. 2) is from Lemma 3.6. □

Theorem 3.8. *IOpen proves $L_{\varphi(x)}$ for any open formula $\varphi(x)$.*

Proof. By Main Lemma 3.7, there exists an n such that $IOpen \vdash \mathcal{I}_n(\varphi(x))$. By induction on n. For $n = 0$, we work in $IOpen$. Assume that $\exists x\varphi(x)$. Since $\mathcal{I}_0(\varphi(x))$ is $\forall x\varphi(x) \vee \forall x\neg\varphi(x)$, we obtain $\forall x\varphi(x)$. We have $\forall y < 0\neg\varphi(y) \wedge \varphi(0)$. Hence $\exists x(\forall y < x\neg\varphi(y) \wedge \varphi(x))$. Next, assume that the assertion holds for $n = 2m$. Recall that $\mathcal{I}_{2m+1}(\varphi(x)) \equiv \mathcal{I}_{2m}(\varphi(x)) \vee \theta_{2m+1}(\varphi(x)) \vee \theta'_{2m+1}(\varphi(x))$. If $\mathcal{I}_{2m}(\varphi(x))$ holds, we have the assertion by the induction hypothesis. In the other case, there exist $a_0, a_1, \cdots, a_{2m}$ such that $a_0 < a_1 < \cdots < a_{2m}$, and $\forall y(\varphi(y) \leftrightarrow y < a_0 \vee (\bigvee_{i=0}^{m-1} a_{2i+1} \leq y < a_{2i+2}))$ or $\forall y(\varphi(y) \leftrightarrow (\bigvee_{i=0}^{m} a_{2i} \leq y < a_{2i+1}) \vee a_{2m} \leq y)$. In the first case $\forall y < 0\neg\varphi(y) \wedge \varphi(0)$, and in the second case $\forall y < a_0\neg\varphi(y) \wedge \varphi(a_0)$. In either case, we have $\exists x(\forall y < x\neg\varphi(y) \wedge \varphi(x))$.

Assume that the assertion holds for $n = 2m + 1$. We can show similarly that the assertion holds for $n = 2m + 2$. □

Corollary 3.9. *IOpen is equivalent to LOpen.*

Proof. It is easy to see that $PA^- \vdash L_{\neg\varphi(x)} \to I_{\varphi(x)}$ for any open formula $\varphi(x)$, hence *LOpen* proves all axioms of *IOpen*. Thus the assertion is immediately obtained from Theorem 3.8. □

We have used open induction only in showing that $\mathcal{I}_n(\varphi(x))$ for an inequality $\varphi(x)$ of degree n. For any n, $\mathcal{I}_n(\varphi(x))$ can not be proved in PA^-. To see this, consider $\mathcal{M} = \{f \in Z[t] \mid 0 \leq LC(f)\}$, where $Z[t]$ is the polynomial ring over the integers and $LC(f)$ is the leading coefficient of f. $\mathcal{M}$ is made into an $\mathcal{L}_A$-structure by defining an order $<$ making t 'infinitely large'. More specifically, if $f, g \in Z[t]$, then we define $g < f \leftrightarrow LC(f-g) > 0$. $\mathcal{M}$ is a model of PA^-, but not of *IOpen*; let $\varphi(x,y) \equiv x^2 < y$ and consider $\varphi(x,t) \equiv x^2 < t$, then $\varphi(0,t) \wedge \exists x \neg\varphi(x,t)$ but there does not exist an element x in $\mathcal{M}$ such that $\varphi(x,t)$ and $\neg\varphi(x+1,t)$. Now assume that $\mathcal{I}_2(\varphi(x,t))$. Since $\forall n \in N \varphi(n,t)$ and $\neg\varphi(t,t)$, $\varphi(x,t)$ must be equivalent to $x < a$ or $x < a \vee b \leq x$ for some $a, b \in \mathcal{M}$ with $0 < a < b$ by Corollary 3.4. In either case we have $\varphi(a-1,t) \wedge \neg\varphi(a,t)$, which is a contradiction.

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